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STEADY CONDUCTION OF HEAT IN PROLATE SPHEROIDAL SYSTEMS

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The steady local temperature distribution has been determined for arbitrary prolate spheroidal systems with various boundary conditions. The temperature was derived for all possible systems, such as when the temperature was kept constant on the hyperbolic surfaces and variable on the prolate spheroidal surfaces, and vice versa. In the latter case, roots with respect to the degree of the associated Legendre function for arguments larger than unity had to be determined. Some of the presented solutions have been evaluated numerically, and the isothermal surfaces have been presented.

In engineering thermal problems, where heat passes through a substance of a body, the treatment for the determination of the local temperature distribution may be performed by considering only the conduction of heat through the material, the effects of convection and radiation being neglected. There are many configurations in aeronautical and aerospace engineering which exhibit prolate spheroidal configuration, such as may be found in airplanes, reentry vehicles, space capsules and space planes.

The conduction of heat has been treated extensively for rectangular, cylindrical and spherical coordinates by Carlslaw and Jaeger [1], but only little information has been given for other shapes and coordinate systems, or even truncated annular sector geometries of such systems. Conical systems have been treated extensively by Bauer [2], who also investigated paraboloidal systems [3] of various forms. The following investigation uses prolate spheroid coordinates (Fig. 1) and determines the local temperature distribution in the prolate spheroidal systems exhibited in Fig. 2. A few special cases have been evaluated numerically.

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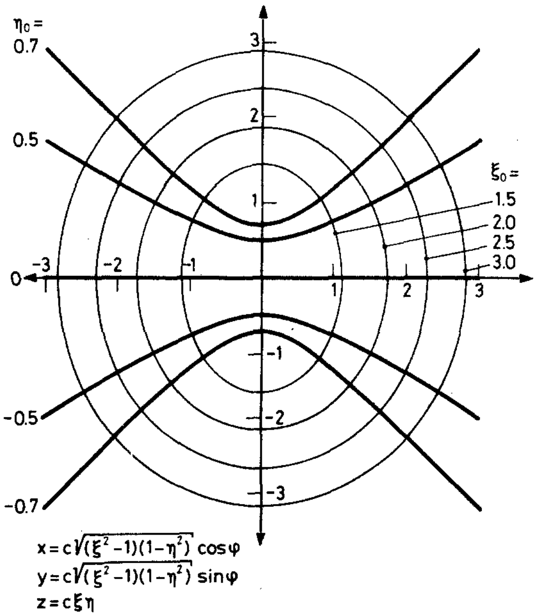


Fig. 1 Geometry and coordinate system

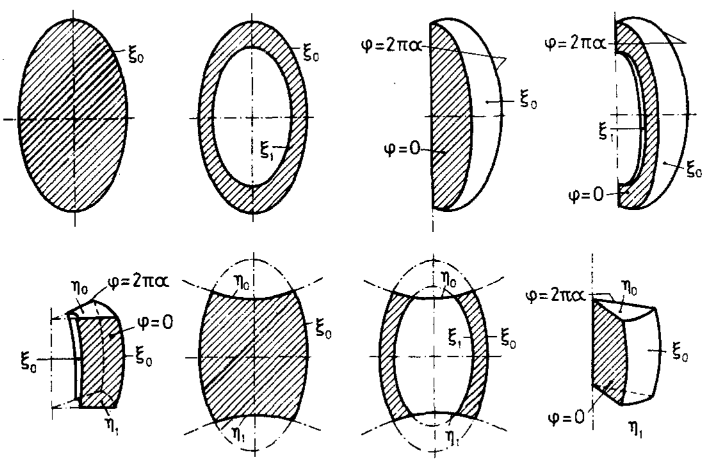


Fig. 2 Special forms of prolate spheroidal systems

Nomenclature

$L^{m}_{\frac{1}{2}\alpha\, \lambda_{mn}}$	see definition (6) for argument ξ, or definition (27) for argument η
$P^{m}_{\frac{1}{2}\alpha\, \lambda_{mn}}$, $Q^{m}_{\frac{1}{2}\alpha\, \lambda_{mn}}$	associated Legendre functions of first and second kind (for argument η the range is $-1 \leqslant \eta_1 \leqslant \eta \leqslant \eta_0 < 1$; for argument ξ it is $1 \leqslant \xi_1 \leqslant \xi \leqslant \xi_0 < \infty$)
T	temperature
$2\pi\alpha$	sector angle
ξ, η, φ	prolate spheroid coordinates
ξ_0, ξ_1	prolate spheroidal surfaces
η_0, η_1	hyperboloid surfaces
λ_{mn}	root of determinants (7), (20) or (26)

Basic equation and solution

For determination of the local temperature in a prolate annular sector spheroid, the equation of conduction in prolate spheroidal coordinates [4] (Fig. 1)

$$\frac{\partial}{\partial \xi}\left[(\xi^2-1)\frac{\partial T}{\partial \xi}\right]+\frac{\partial}{\partial \eta}\left[(1-\eta^2)\frac{\partial T}{\partial \eta}\right]+\frac{(\xi^2-\eta^2)}{(1-\eta^2)(\xi^2-1)}\frac{\partial^2 T}{\partial \varphi^2}=0 \quad (1)$$

must be solved with the appropriate boundary conditions. This equation is solved by the method of separation, giving solutions of the form (because of the chosen coordinate relations (see Fig. 1)

$$T(\xi, \eta, \varphi) = \{AP^{\mu}_{\lambda}(\eta)+BQ^{\mu}_{\lambda}(\eta)\}\{CP^{\mu}_{\lambda}(\xi)+DQ^{\mu}_{\lambda}(\xi)\}\{E\cos\mu\varphi+F\sin\mu\varphi\} \quad (2)$$

where $-1 \leqslant \eta \leqslant +1$; $1 \leqslant \xi < \infty$ and $0 \leqslant \varphi \leqslant 2\pi$. This is an eigenvalue problem, in which the eigenvalues λ_{mn} have to be determined. For a truncated prolate annular sector spheroid, the boundary conditions are

$$T = T_0 \quad \text{for} \quad \varphi = 0, 2\pi\alpha \quad \text{and for} \quad \xi = \xi_0, \xi_1, \quad (3)$$

and

$$\left.\begin{aligned} T &= g_0(\xi, \varphi) \quad \text{at} \quad \eta = \eta_0 \\ T &= g_1(\xi, \varphi) \quad \text{at} \quad \eta = \eta_1 \end{aligned}\right\} \quad (4)$$

For compatibility, the boundaries of the planes $\eta = \eta_0$ and η_1 should exhibit $T = T_0$, i.e. $g_j(\xi, 0) = g_j(\xi, 2\pi\alpha) = g_j(\xi_0, \varphi) = g_j(\xi_1, \varphi) = T_0$ $(j = 0, 1)$.

The solution of the equation of heat conduction is given by

$$T(\xi,\eta,\varphi) = T_0 + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\{A_{mn}P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta) + B_{mn}Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta)\right\} L^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi)\sin\tfrac{m}{2\alpha}\varphi \tag{5}$$

where

$$L^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi) \equiv P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi)Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi_0) - P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi_0)Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi) \tag{6}$$

and the roots λ_{mn} are obtained from the determinant $(\xi_0 > \xi_1 \geqslant 1)$

$$\begin{vmatrix} P^{\frac{m}{2\alpha}}_{\lambda}(\xi_0) & Q^{\frac{m}{2\alpha}}_{\lambda}(\xi_0) \\ P^{\frac{m}{2\alpha}}_{\lambda}(\xi_1) & Q^{\frac{m}{2\alpha}}_{\lambda}(\xi_1) \end{vmatrix} = 0 \tag{7}$$

Solution (5) satisfies the boundary conditions in φ and ξ (see Eq. (3)). In order to determine the remaining integration constants, the functions g_0 and g_1 have to be expanded into Legendre–Fourier-sine series*, i.e.

$$g_1(\xi,\varphi) - T_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn}L^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi)\sin\tfrac{m}{2\alpha}\varphi \tag{8a}$$

and

$$g_2(\xi,\varphi) - T_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta_{mn}L^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi)\sin\tfrac{m}{2\alpha}\varphi \tag{8b}$$

which, with the boundary conditions (4), yields

$$A_{mn} = \frac{\alpha_{mn}Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_1) - \beta_{mn}Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_0)}{\left\{P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_0)Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_1) - P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_1)Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_0)\right\}} \tag{9a}$$

and

$$B_{mn} = \frac{\beta_{mn}P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_0) - \alpha_{mn}P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_1)}{\left\{P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_0)Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_1) - P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_1)Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta_0)\right\}} \tag{9b}$$

The expansion coefficients α_{mn} and β_{mn} are obtained with the orthogonality relation of $L^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi)$, i.e.

* It may be shown that, for a function $f(\eta,\varphi)$ which is absolutely integrable over the region $0 \leqslant \varphi \leqslant 2\pi$, $-1 \leqslant \eta \leqslant +1$, solution (5) in the finite domain with $\xi_0 \leqslant \xi \leqslant \xi_1$ converges on the surface of the body to $f(\eta,\varphi)$ (see [5]). It is known that the solution of this problem is unique. The potential function for the internal space, i.e. $1 \leqslant \xi \leqslant \xi_0$, would be represented by replacing $L^{\frac{m}{2\alpha}}_{\lambda_{mn}}$ with $P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi)$, since $Q^{\frac{m}{2\alpha}}_{\lambda}(\xi)$ becomes infinite at $\xi = 1$. The exterior space $\xi > \xi_0$ yields a potential function in which $L^{m/2}_{\lambda_{mn}}$ is replaced by $Q^{\frac{m}{2\alpha}}_{\lambda_m}(\xi)$, since $P^{\frac{m}{2\alpha}}_{\lambda}(\xi)$ becomes infinite for $\xi \to \infty$.

$$\int_{\xi_1}^{\xi_0} L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi) L_{\lambda_{mv}}^{\frac{m}{2\alpha}}(\xi)\, d\xi = \begin{cases} 0 \quad \text{for} \quad v \neq n \\ \dfrac{(1-\xi_0^2)\dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_0)}{\partial \lambda_{mn}} \dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_0)}{\partial \xi} - (1-\xi_1^2)\dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_1)}{\partial \lambda_{mn}} \dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_1)}{\partial \xi}}{(2\lambda_{mn}+1)} \\ \qquad \text{for } v = n \end{cases} \tag{10}$$

(These results are obtained in the usual way by multiplying the governing Legendre differential equation for n by $L_{\lambda_{mv}}^{\frac{m}{2\alpha}}$, that for v by $L_{\lambda_{mn}}^{\frac{m}{2\alpha}}$, subtracting them from each other and integrating with respect to ξ from ξ_0 to ξ_1. This yields

$$\begin{Bmatrix} \alpha_{mn} \\ \beta_{mn} \end{Bmatrix} = \frac{(2\lambda_{mn}+1) \int_0^{2\pi\alpha} \int_{\xi_0}^{\xi_0} \begin{Bmatrix} g_0(\xi,\varphi) - T_0 \\ g_1(\xi,\varphi) - T_0 \end{Bmatrix} L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi) \sin \frac{m}{2\alpha}\varphi \, d\xi\, d\varphi}{\left\{(1-\xi_1^2)\dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_1)}{\partial \lambda_{mn}} \cdot \dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_1)}{\partial \xi} - (1-\xi_0^2)\dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_0)}{\partial \lambda_{mn}} \cdot \dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_0)}{\partial \xi} \right\} \pi\alpha} \tag{11}$$

Introduction of these results into Eq. (5) gives the solution for the local temperature distribution.

If at the surfaces $\varphi = 0$, $2\pi\alpha$ and $\xi = \xi_0, \xi_1$ the system exhibits no flux across them, with the boundary conditions (4) and

$$\frac{\partial T}{\partial \varphi} = 0 \quad \text{at} \quad \varphi = 0, 2\pi\alpha \quad \text{and} \quad \frac{\partial T}{\partial \xi} = 0 \quad \text{at} \quad \xi = \xi_0, \xi_1 \tag{12}$$

the solution for the temperature yields the expression

$$T(\xi,\eta,\varphi) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \{A'_{mn} P_{\lambda'_{mn}}^{\frac{m}{2\alpha}}(\eta) + B'_{mn} Q_{\lambda'_{mn}}^{\frac{m}{2\alpha}}(\eta)\} L_{\lambda'_{mn}}^{\frac{m}{2\alpha}}(\xi) \cos \frac{m}{2\alpha}\varphi \tag{13}$$

where λ'_{mn} are the roots of

$$\begin{vmatrix} P'^{\frac{m}{2\alpha}}_{\lambda'}(\xi_0) & Q'^{\frac{m}{2\alpha}}_{\lambda'}(\xi_0) \\ P'^{\frac{m}{2\alpha}}_{\lambda'}(\xi_1) & Q'^{\frac{m}{2\alpha}}_{\lambda'}(\xi_1) \end{vmatrix} = 0 \tag{14}$$

Expansion of $g_i(\xi,\varphi)$ into Legendre–Fourier-cosine series

$$g_i(\xi,\varphi) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \begin{Bmatrix} \alpha'_{mn} \\ \beta'_{mn} \end{Bmatrix} L_{\lambda'_{mn}}^{\frac{m}{2\alpha}}(\xi) \cos \frac{m}{2\alpha}\varphi \quad (i = 0, 1) \tag{15}$$

with the orthogonality relation of $L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi)$

$$\int_{\xi_1}^{\xi_0} L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi) L^{\frac{m}{2\alpha}}_{\lambda'_{m\nu}}(\xi)\,d\xi = \begin{cases} 0 \quad \text{for } \nu \neq n & (16) \\ \dfrac{(1-\xi_0^2)\dfrac{\partial^2 L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi_0)}{\partial\xi\partial\lambda'_{mn}} L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi_0) - (1-\xi_1^2)\dfrac{\partial^2 L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi_1)}{\partial\xi\partial\lambda'_{mn}} L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi_1)}{(2\lambda'_{mn}+1)} & \text{for } \nu = n \end{cases}$$

yields the integration constants A'_{mn} and B'_{mn} from Eq. (9a) and (9b) by writing prime values for all λ, α and β. The expressions

$$\begin{Bmatrix} \alpha'_{mn} \\ \beta'_{mn} \end{Bmatrix} = \frac{(2\lambda'_{mn}+1)\displaystyle\int_0^{2\pi\alpha}\int_{\xi_1}^{\xi_0} \begin{Bmatrix} g_0(\xi,\varphi) \\ g_1(\xi,\varphi) \end{Bmatrix} L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi)\cos\tfrac{m}{2\alpha}\varphi\,d\xi\,d\varphi}{\left\{(1-\xi_0^2) L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi_0)\dfrac{\partial^2 L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi_0)}{\partial\xi\partial\lambda'_{mn}} - (1-\xi_1^2) L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi_1)\dfrac{\partial^2 L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi_1)}{\partial\xi\partial\lambda'_{mn}}\right\}\pi\alpha} \tag{17}$$

have to be introduced there. Finally, after the introduction of A'_{mn} and B'_{mn} into Eq. (13), the local temperature distribution is obtained.

If the wall at $\xi=\xi_0$ exhibits no flux:

$$\frac{\partial T}{\partial \xi} = 0 \quad \text{at} \quad \xi = \xi_0 \tag{18}$$

and the wall at $\xi = \xi_1$:

$$T = T_0 \quad \text{at} \quad \xi = \xi_1 \tag{19}$$

then the roots $\bar{\lambda}_{mn}$ are obtained from

$$\begin{vmatrix} P'^{\frac{m}{2\alpha}}_{\bar\lambda}(\xi_0) & Q'^{\frac{m}{2\alpha}}_{\bar\lambda}(\xi_0) \\ P^{\frac{m}{2\alpha}}_{\bar\lambda}(\xi_1) & Q^{\frac{m}{2\alpha}}_{\bar\lambda}(\xi_1) \end{vmatrix} = 0 \tag{20}$$

while the integration constants $\bar{A}_{mn}$ and $\bar{B}_{mn}$ are obtained from Eq. (9) for $\bar{\lambda}_{mn}$ and $\bar{\alpha}_{mn}$, $\bar{\beta}_{mn}$. The latter, with the orthogonality relation

$$\int_{\xi_1}^{\xi_0} L^{\frac{m}{2\alpha}}_{\bar\lambda_{mn}}(\xi) L^{\frac{m}{2\alpha}}_{\bar\lambda_{mn}}(\xi)\,d\xi = \begin{cases} 0 \quad \text{for } n \neq \nu \\ \dfrac{-\left\{(1-\xi_0^2)\cdot\dfrac{\partial^2 L^{\frac{m}{2\alpha}}_{\bar\lambda_{mn}}(\xi_0)}{\partial\xi\partial\bar\lambda_{mn}}\cdot L^{\frac{m}{2\alpha}}_{\bar\lambda_{mn}}(\xi_0) + (1-\xi_1^2)\dfrac{\partial L^{\frac{m}{2\alpha}}_{\bar\lambda_{mn}}(\xi_1)}{\partial\xi}\cdot\dfrac{\partial L^{\frac{m}{2\alpha}}_{\bar\lambda_{mn}}}{\partial\bar\lambda_{mn}}\right\}}{(2\bar\lambda_{mn}+1)} & \text{for } \nu = n \end{cases} \tag{21}$$

are given by

$$\begin{Bmatrix} \bar{\alpha}_{mn} \\ \bar{\beta}_{mn} \end{Bmatrix} = - \frac{(2\bar{\lambda}_{mn}+1) \int\limits_{0}^{2\pi\alpha} \int\limits_{\xi_1}^{\xi_0} \begin{Bmatrix} g_0(\xi, \varphi) \\ g_1(\xi, \varphi) - T_0 \end{Bmatrix} L_{\bar{\lambda}_{mn}}^{\frac{m}{2\alpha}}(\xi) \cos(\tfrac{m}{2\alpha}\varphi)\, d\xi\, d\varphi}{\pi\alpha \left\{ (1-\xi_0^2) \frac{\partial^2 L_{\bar{\lambda}_{mn}}^{\frac{m}{2\alpha}}(\xi_0)}{\partial\xi \partial\bar{\lambda}_{mn}} L_{\bar{\lambda}_{mn}}^{\frac{m}{2\alpha}}(\xi_0) + (1-\xi_1^2) \frac{\partial L_{\bar{\lambda}_{mn}}^{\frac{m}{2\alpha}}(\xi_1)}{\partial \bar{\lambda}_{mn}} \cdot \frac{\partial L_{\bar{\lambda}_{mn}}^{\frac{m}{2\alpha}}(\xi_1)}{\partial\xi} \right\}} \tag{22}$$

If the boundary $\varphi = 0, 2\pi\alpha$ exhibits $T = T_0$, then $\cos(\frac{m}{2\alpha}\varphi)$ must be replaced by $\sin(\frac{m}{2\alpha}\varphi)$. If the boundary condition at $\xi = \xi_0$ is $T = T_0$ and that at $\xi = \xi_1$ is $\frac{\partial T}{\partial \xi} = 0$, then the λ_{mn}^* values are obtained from determinant (20), in which ξ_0 and ξ_1 are exchanged. The α_{mn}^* and β_{mn}^* values are obtained from Eq. (22) by writing for $\bar{\lambda}$ the values λ_{mn}^*, and instead of g_0 the expression $g_0(\xi, \varphi) - T_0$, while for $g_1 - T_0$ we use $g_1(\xi, \varphi)$. In the denominator the values ξ_0 and ξ_1 are exchanged and the minus sign in front of the total expression must be changed into a plus sign. With these results, the constants A_{mn}^* and β_{mn}^* may be obtained from Eq. (9). Thus, the local temperature distribution is then presented by Eq. (5), where $\sin\frac{m}{2\alpha}\varphi$ has been replaced by $\cos\frac{m}{2\alpha}\varphi$ according to the given boundary conditions at $\varphi = 0, 2\pi\alpha$.

The temperature distribution in a prolate annular sector spheroid with the boundary conditions

$$T = T_0 \quad \text{at} \quad \varphi = 0, 2\pi\alpha \quad \text{and} \quad T = T_0 \quad \text{at} \quad \eta = \eta_1, \eta_0 \tag{23}$$

and observing the compatibility of the boundaries, i.e. $T = T_0$ on the planes $\xi = \xi_0$ and $\xi = \xi_1$ $(f_j(\eta, 0) = f_j(\eta, 2\pi\alpha) = f_j(\eta_0, \varphi) = f_j(\eta_1, \varphi) = T_0, j = 0, 1)$

$$\begin{aligned} T &= f_0(\eta, \varphi) \quad \text{at} \quad \xi = \xi_0 \\ T &= f_1(\eta, \varphi) \quad \text{at} \quad \xi = \xi_1 \end{aligned} \tag{24}$$

yields for the heat conduction Eq. (1) a solution of the form

$$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[C_{mn} P_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi) + D_{mn} Q_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi) \right] L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta) \sin\tfrac{m}{2\alpha}\varphi \tag{25}$$

where λ_{mn} are the roots of the determinant $(\eta_0 < \eta_1 < 1) - 1 < \eta_0 < \eta_1 < 1$ [6–8]

$$\begin{vmatrix} P_{\lambda}^{\frac{m}{2\alpha}}(\eta_0) & Q_{\lambda}^{\frac{m}{2\alpha}}(\eta_0) \\ P_{\lambda}^{\frac{m}{2\alpha}}(\eta_1) & Q_{\lambda}^{\frac{m}{2\alpha}}(\eta_1) \end{vmatrix} = 0 \tag{26}$$

and $L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta)$ is defined as

$$L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta) = P_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta)Q_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta_0) - P_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta_0)Q_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta) \tag{27}$$

Equation (25) satisfies the boundary conditions (23). It may be noted here that the argument η of the above associated Legendre functions is absolutely smaller than unity, while the ξ values of the previous case (see Eq. (7)) are larger than unity.

To satisfy the remaining boundary conditions (24), the functions f_i $(i = 0, 1)$ must be expanded in Legendre–Fourier-sine series, such that $(i = 0, 1)$

$$f_i(\eta, \varphi) - T_0 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \begin{Bmatrix} \gamma_{mn} \\ \delta_{mn} \end{Bmatrix} L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta) \sin\left(\tfrac{m}{2\alpha}\varphi\right) \tag{28}$$

This may be achieved with the orthogonality relation of the Legendre functions $L_{\lambda}^{\frac{m}{2\alpha}}(\eta)$, which are given by

$$\int_{\eta_1}^{\eta_0} L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta)\, L_{\lambda_{m\nu}}^{\frac{m}{2\alpha}}(\eta)\, d\eta = \begin{cases} 0 \quad \text{for} \quad \nu \neq n \\[2ex] \dfrac{(1-\eta_0^2)\dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta_0)}{\partial \lambda_{mn}} \dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta_0)}{\partial \eta} - (1-\eta_1^2)\dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta_1)}{\partial \lambda_{mn}} \dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta_1)}{\partial \eta}}{(2\lambda_{mn}+1)} \quad \text{for} \quad \nu = n \end{cases} \tag{29}$$

The expansion coefficients γ_{mn} and δ_{mn} are then

$$\begin{Bmatrix} \gamma_{mn} \\ \delta_{mn} \end{Bmatrix} = \frac{(2\lambda_{mn}+1) \displaystyle\int_0^{2\pi\alpha} \int_{\eta_1}^{\eta_0} \begin{Bmatrix} f_0(\eta, \varphi) - T_0 \\ f_1(\eta, \varphi) - T_0 \end{Bmatrix} L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta) \sin\left(\tfrac{m}{2\alpha}\varphi\right) d\varphi\, d\eta}{\pi\alpha \left\{ (1-\eta_0^2)\dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta_0)}{\partial \lambda_{mn}} \dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta_0)}{\partial \eta} - (1-\eta_1^2)\dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta_1)}{\partial \lambda_{mn}} \dfrac{\partial L_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\eta_1)}{\partial \eta} \right\}} \tag{30}$$

With these results, the integration constants C_{mn} and D_{mn} are given by

$$C_{mn} = \frac{\gamma_{mn} Q_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_1) - \delta_{mn} Q_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_0)}{\left[P_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_0) Q_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_1) - P_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_1) Q_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_0) \right]} \tag{31a}$$

$$D_{mn} = \frac{\delta_{mn} P_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_0) - \gamma_{mn} P_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_1)}{\left[P_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_0) Q_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_1) - P_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_1) Q_{\lambda_{mn}}^{\frac{m}{2\alpha}}(\xi_0) \right]} \tag{31b}$$

Introduction of these into Eq. (25) gives the local temperature distribution. In a similar fashion as in the previous cases, all combinations of boundary conditions of constant temperature or of no flux across the boundary may be obtained. In these cases, λ assumes the values λ', $\bar{\lambda}$ and λ^*, respectively, and is obtained from the appropriate determinant. It may be emphasized, however, that the associated Legendre functions in η are to be considered in the range $-1 \leqslant \eta \leqslant +1$. i.e. in the case of an annular system in the range $\eta_1 \leqslant \eta \leqslant \eta_0$. The associated Legendre functions with the argument ξ range in the interval $1 < \xi < \infty$, which in the above particular cases satisfy $\xi_1 \leqslant \xi \leqslant \xi_0$ within that range. The mentioned values of λ', $\bar{\lambda}$ and λ^* relate to the boundary conditions of no flux across $\eta = \eta_0$ and η_1, and no flux across $\eta = \eta_1$, respectively, and are obtained from the determinants [6, 9, 10]

$$\begin{vmatrix} P'^{\frac{m}{2\alpha}}_{\lambda'}(\eta_0) & Q'^{\frac{m}{2\alpha}}_{\lambda'}(\eta_0) \\ P'^{\frac{m}{2\alpha}}_{\lambda'}(\eta_1) & Q'^{\frac{m}{2\alpha}}_{\lambda'}(\eta_1) \end{vmatrix} = 0, \quad \begin{vmatrix} P'^{\frac{m}{2\alpha}}_{\bar{\lambda}}(\eta_0) & Q'^{\frac{m}{2\alpha}}_{\bar{\lambda}}(\eta_0) \\ P^{\frac{m}{2\alpha}}_{\bar{\lambda}}(\eta_1) & Q^{\frac{m}{2\alpha}}_{\bar{\lambda}}(\eta_1) \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} P^{\frac{m}{2\alpha}}_{\lambda^*}(\eta_0) & Q^{\frac{m}{2\alpha}}_{\lambda^*}(\eta_0) \\ P'^{\frac{m}{2\alpha}}_{\lambda^*}(\eta_1) & Q'^{\frac{m}{2\alpha}}_{\lambda^*}(\eta_1) \end{vmatrix} = 0$$

respectively. The local temperature distribution may therefore be obtained in an analogous fashion as in the previous cases, by observing the appropriate orthogonality relation. For no flux across the walls $\eta = \eta_0$ and η_1, it is given by

$$\int_{\eta_1}^{\eta_0} L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\eta)\, L^{\frac{m}{2\alpha}}_{\lambda'_{m\nu}}(\eta)\, d\eta = \begin{cases} 0 \quad \text{for} \quad \nu \neq n & (32) \\ \dfrac{(1-\eta_0^2)\dfrac{\partial^2 L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\eta_0)}{\partial \lambda'_{mn} \partial \eta} L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\eta_0) - (1-\eta_1^2)\dfrac{\partial^2 L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\eta_1)}{\partial \lambda'_{mn} \partial \eta} L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\eta_0)}{(2\lambda'_{mn}+1)} & \text{for} \quad \nu = n \end{cases}$$

while for no flux across $\eta = \eta_0$ and constant temperature at $\eta = \eta_1$, it is

$$\int_{\eta_1}^{\eta_0} L^{\frac{m}{2\alpha}}_{\bar{\lambda}_{mn}}(\eta)\, L^{\frac{m}{2\alpha}}_{\bar{\lambda}_{mn}}(\eta)\, d\eta = \begin{cases} 0 \quad \text{for} \quad \nu \neq n & (33) \\ \dfrac{\left\{(1-\eta_0^2)\dfrac{\partial^2 L^{\frac{m}{2\alpha}}_{\bar{\lambda}_{mn}}(\eta_0)}{\partial \bar{\lambda}_{mn} \partial \eta} L^{\frac{m}{2\alpha}}_{\bar{\lambda}_{mn}}(\eta_0) + (1-\eta_1^2)\dfrac{\partial L^{\frac{m}{2\alpha}}_{\bar{\lambda}_{mn}}(\eta_1)}{\partial \lambda_{mn}} \cdot \dfrac{\partial L^{\frac{m}{2\alpha}}_{\bar{\lambda}_{mn}}(\eta_1)}{\partial \eta}\right.}{(2\bar{\lambda}_{mn}+1)} & \text{for} \quad \nu = n \end{cases}$$

In the case of constant temperature at $\eta=\eta_0$ and no flux at $\eta=\eta_1$, the eigenvalue $\bar{\lambda}$ is substituted by λ^*, η_0 and η_1 are exchanged and the minus sign in the previous orthogonality expression is omitted.

In the case of a *prolate spheroidal sector part* geometry, with the boundary conditions (observing the compatibility conditions of the planes $\eta=\eta_0$ and η_1)

$$T=T_0 \text{ at } \xi=\xi_0 \text{ and } \varphi=0,2\pi\alpha,\ T=g_0(\xi,\varphi) \text{ at } \eta=\eta_0 \text{ and } T=g_1(\xi,\varphi) \text{ at } \eta=\eta_1 \tag{34}$$

the solution Eq. (5) may be used, in which the $L(\xi)$ function must be replaced by $\overset{m}{P}{}_{\lambda_{mn}}^{\frac{\overline{\alpha}}{2}}(\xi)$. The roots λ_{mn} in this case are obtained from

$$\overset{m}{P}{}_{\lambda}^{\frac{\overline{\alpha}}{2}}(\xi_0)=0 \tag{35}$$

where $\lambda_{mn}=-\frac{1}{2}+i\alpha^*_{mn}$ (see Table 2).

The integration constants A_{mn} and B_{mn} may be determined from (9) with α_{mn} and β_{mn} from Eq. (11), where L has been substituted by $\overset{m}{P}{}_{\lambda_{mn}}^{\frac{\overline{\alpha}}{2}}(\xi)$ and the orthogonality relation

$$\int_1^{\xi_0}\overset{m}{P}{}_{\lambda_{mn}}^{\frac{\overline{\alpha}}{2}}(\xi)\overset{m}{P}{}_{\lambda_{mv}}^{\frac{\overline{\alpha}}{2}}(\xi)\,d\xi=\begin{cases}0 & \text{for } n\neq v\\[2ex] \dfrac{(1-\xi_0^2)}{(2\lambda_{mn}+1)}\,\dfrac{\partial\overset{m}{P}{}_{\lambda_{mn}}^{\frac{\overline{\alpha}}{2}}(\xi_0)}{\partial\lambda_{mn}}\cdot\dfrac{\partial\overset{m}{P}{}_{\lambda_{mn}}^{\frac{\overline{\alpha}}{2}}(\xi_0)}{\partial\xi} & \text{for } v=n\end{cases} \tag{36}$$

is observed there for the denominator. For no flux across $\xi=\xi_0$, i.e. $\frac{\partial T}{\partial\xi}=0$ at $\xi=\xi_0$ and $\frac{\partial T}{\partial\varphi}=0$ at $\varphi=0,\ 2\pi\alpha$, the solution is given by expression (13), in which λ' is obtained from $\partial\overset{m}{P}{}_{\lambda'}^{\frac{\overline{\alpha}}{2}}(\eta_0)/\partial\xi=0$ and L has again been substituted by $\overset{m}{P}{}_{\lambda'_{mn}}^{\frac{\overline{2\alpha}}{}}(\xi)$. The orthogonality relation in this case reads

$$\int_1^{\xi_0}\overset{m}{P}{}_{\lambda'_{mn}}^{\frac{\overline{2\alpha}}{}}(\xi)\overset{m}{P}{}_{\lambda'_{mn}}^{\frac{\overline{2\alpha}}{}}(\xi)\,d\xi)=\begin{cases}0 & \text{for } v\neq n\\[2ex] \dfrac{-(1-\xi_0^2)}{(2\lambda'_{mn}+)}\,\dfrac{\partial^2\overset{m}{P}{}_{\lambda'_{mn}}^{\frac{\overline{2\alpha}}{}}(\xi_0)}{\partial\lambda'_{mn}\partial\xi}\,\overset{m}{P}{}_{\lambda'_{mn}}^{\frac{\overline{2\alpha}}{}}(\xi_0) & \text{for } v=n\end{cases} \tag{37}$$

where $\lambda'_{mn}=-\frac{1}{2}+i\beta^*_{mn}$ (see Table 3).

For a *prolate spheroidal sector part* satisfying the boundary conditions

$$T = T_0 \text{ at } \varphi = 0, 2\pi\alpha \text{ and } T = T_0 \text{ at } \eta = \eta_1, \eta_0$$
$$T = f_0(\eta, \varphi) \text{ at } \xi = \xi_0 \tag{38}$$

solution (25) may be employed by omitting $Q^{\frac{m}{2\alpha}}_{\lambda}(\xi)$, i.e. $D_{mn} = 0$.

The magnitude of the integration constant C_{mn} is obtained as $C_{mn} = \gamma_{mn}/P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi_0)$, where γ_{mn} is the expansion coefficient (Eq. (28)) and is presented by (30). If no heat flux occurs across the boundaries $\varphi = 0, 2\pi\alpha$ and $\eta = \eta_0, \eta_1$, the local temperature distribution is given by

$$T(\xi, \eta, \varphi) = \sum_{n=0}^{\infty} \sum_{n=1}^{\infty} C'_{mn} P^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\xi) L^{\frac{m}{2\alpha}}_{\lambda'_{mn}}(\eta) \cos\left(\tfrac{m}{2\alpha}\varphi\right) \tag{39}$$

With the orthogonality relation (32), the constants C'_{mn} may be obtained.

For an *annular spheroidal sector with no η-surfaces,* the boundary conditions

$$T = T_0 \text{ at } \varphi = 0, 2\pi\alpha \text{ and } T = f_0(\eta, \varphi) \text{ at } \xi = \xi_0, \; T = f_1(\eta, \varphi) \text{ at } \xi = \xi_1 \tag{40}$$

must be satisfied. A temperature distribution satisfying the boundary conditions in the φ direction is given by Eq. (25), in which $L^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta)$ is to be substituted by $P^{\frac{m}{2\alpha}}_{n}(\eta)$. It reads

$$T(\xi, \eta, \varphi) = T_0 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [C_{mn} P^{\frac{m}{2\alpha}}_{n}(\xi) + D_{mn} Q^{\frac{m}{2\alpha}}_{n}(\xi)] P^{\frac{m}{2\alpha}}_{n}(\eta) \sin\left(\tfrac{m}{2\alpha}\varphi\right) \tag{41}$$

If the remaining boundary conditions in ξ_0 and ξ_1 are satisfied, the expansion of

$$f_i(\eta, \varphi) - T_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{Bmatrix} \alpha_{mn} \\ \beta_{mn} \end{Bmatrix} P^{\frac{m}{2\alpha}}_{n}(\eta) \sin\left(\tfrac{m}{2\alpha}\varphi\right) \tag{42}$$

and the orthogonality relation of the associated Legendre functions ($m/2\alpha$ integer)

$$\int_{-1}^{+1} P^{\frac{m}{2\alpha}}_{n}(\eta) P^{\frac{m}{2\alpha}}_{\nu}(\eta)\, d\eta = \begin{cases} 0 & \text{for } \nu \neq n \\ \dfrac{2}{(2n+1)} \dfrac{\left(n + \dfrac{m}{2\alpha}\right)!}{\left(n - \dfrac{m}{2\alpha}\right)!} & \text{for } \nu = n \end{cases} \tag{43}$$

yields the integration constants C_{mn} and D_{mn}. They are given by

$$C_{mn} = \frac{\alpha_{mn} Q_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_1) - \beta_{mn} Q_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_0)}{\left[P_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_0) Q_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_1) - P_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_1) Q_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_0) \right]}$$

and

$$D_{mn} = \frac{\beta_{mn} P_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_0) - \alpha_{mn} P_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_1)}{\left[P_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_0) Q_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_1) - P_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_1) Q_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_0) \right]}$$

where α_{mn} and β_{mn} are the expansion coefficients (42) and are given by

$$\begin{Bmatrix} \alpha_{mn} \\ \beta_{mn} \end{Bmatrix} = \frac{(2n+1)\left(n-\frac{m}{2\alpha}\right)!}{2\left(n+\frac{m}{2\alpha}\right)!\,\pi\alpha} \int\limits_0^{2\pi\alpha} \int\limits_{-1}^{+1} \begin{Bmatrix} f_0(\eta,\varphi)-T_0 \\ f_1(\eta,\varphi)-T_0 \end{Bmatrix} P_{\hat{n}}^{\frac{m}{2\alpha}}(\eta) \sin \tfrac{m}{2\alpha}\varphi \, d\varphi \, d\eta \quad (44)$$

For no flux across the side walls $\varphi = 0, 2\pi\alpha$, $\frac{\partial T}{\partial \varphi} = 0$ and the $\sin \frac{m}{2\alpha}\varphi$ term must be substituted by $\cos \frac{m}{2\alpha}\varphi$ and T_0 must be set equal to zero.

A *prolate spheroidal sector* with the boundary conditions $T = T_0$ at $\varphi = 0, 2\pi\alpha$ and $T = f_0(\eta, \varphi)$ at $\xi = \xi_0$ exhibits a temperature distribution of the form

$$T(\xi,\eta,\varphi) = T_0 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} P_{\hat{n}}^{\frac{m}{2\alpha}}(\xi) P_{\hat{n}}^{\frac{m}{2\alpha}}(\eta) \sin\left(\tfrac{m}{2\alpha}\varphi\right) \quad (45)$$

With the expansion of $f_0(\eta, \varphi) - T_0$ into a Legendre–Fourier-sine series (Eq. (42)), with the orthogonality relation (49) for the integration constants A_{mn} we obtain expression

$$A_{mn} = \alpha_{mn} / P_{\hat{n}}^{\frac{m}{2\alpha}}(\xi_0)$$

For no flux across $\varphi = 0, 2\pi\alpha$, i.e. $\frac{\partial T}{\partial \varphi} = 0$, the function $\sin\left(\frac{m}{2\alpha}\varphi\right)$ must be substituted by $\cos\left(\frac{m}{2\alpha}\varphi\right)$ and T_0 must be taken equal to zero.

The various solutions are represented in Table 1. For the cases of no flux across the boundaries $\varphi = 0, 2\pi\alpha$, $\sin\left(\frac{m}{2\alpha}\varphi\right)$ must be replaced by $\cos\left(\frac{m}{2\alpha}\varphi\right)$, while for the other surfaces the above results should be observed.

Table 1 Analytical expression for solutions of different prolate spheroid geometries

Case	Boundary conditions	Range of variables (domain)	Form of Solution	Geometry
I	$T = T_0 + f_0(\eta, \varphi)$ at $\xi = \xi_0$	$1 \leqslant \xi \leqslant \xi_0$ $-1 \leqslant \eta \leqslant +1$ $0 \leqslant \varphi < 2\pi$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} P_n^m(\xi) P_n^m(\eta) \cos m\varphi$ (example 1)	spheroid
II	$T = T_0 + f_j(\eta, \varphi)$ at $\xi = \xi_i$ $(i = 0, 1)$	$\xi_1 \leqslant \xi \leqslant \xi_0$ $-1 \leqslant \eta \leqslant 1$ $0 \leqslant \varphi < 2\pi$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [A_{mn} P_n^m(\xi) + B_{mn} Q_n^m(\xi)] P_n^m(\eta) \cos m\varphi$	annular spheroid
III	$T = T_0$ at $\varphi = 0, 2\pi\alpha$ $T = T_0 + f_0(\eta, \varphi)$ at $\xi = \xi_0$	$1 \leqslant \xi \leqslant \xi_0$ $-1 \leqslant \eta \leqslant +1$ $0 \leqslant \varphi \leqslant 2\pi\alpha$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} P_n^{m/2\alpha}(\xi) P_n^{m/2\alpha}(\eta) \sin(\frac{m}{2\alpha} \varphi)$	sector spheroid
IV	$T = T_0$ at $\varphi = 0, 2\pi\alpha$ $T = T_0 + f_i(\eta, \varphi)$ at $\xi = \xi_i$ $(i = 0, 1)$	$\xi_1 \leqslant \xi \leqslant \xi_0$ $-1 \leqslant \eta \leqslant +1$ $0 \leqslant \varphi \leqslant 2\pi\alpha$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [A_{mn} P_n^{m/2\alpha}(\xi) + B_{mn} Q_n^{m/2\alpha}(\xi) \sin(\frac{m}{2\alpha} \varphi)$	annular sector spheroid
V	$T = T_0$ at $\eta = \eta_0, \eta_1$ and $\varphi = 0, 2\pi\alpha$ $T = T_0 + f_i(\eta, \varphi)$ at $\xi = \xi_i$ $(i = 0, 1)$	$\xi_1 \leqslant \xi \leqslant \xi_0$ $\eta_1 \leqslant \eta \leqslant \eta_0$ $0 \leqslant \varphi \leqslant 2\pi\alpha$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [A_{mn} P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi) + B_{mn} Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi)] L^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta) \sin(\frac{m}{2\alpha} \varphi)$ λ_{mn} from $L^{\frac{m}{2\alpha}}_{\lambda}(\eta_1) = 0$ (26) (example 2)	truncated annular sector spheroid
VI	$T = T_0$ at $\varphi = 0, 2\pi\alpha$ and $\xi = \xi_0, \xi_1$ $T = T_0 + g_i(\xi, \varphi)$ at $\eta = \eta_i$ $(i = 0, 1)$	$\xi_1 \leqslant \xi \leqslant \xi_0$ $\eta_1 \leqslant \eta \leqslant \eta_0$ $0 \leqslant \varphi \leqslant 2\pi\alpha$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [A_{mn} P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta) + B_{mn} Q^{\frac{m}{2\alpha}}_{\lambda}(\eta)] L^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi) \sin(\frac{m}{2\alpha} \varphi)$ λ_{mn} from $L^{\frac{m}{2\alpha}}_{\lambda}(\xi_1) = 0$ (7) (example 3)	truncated annular spheroid

(Table 1. cont.)

Case	Boundary conditions	Range of variables (domain)	Form of solution	Geometry
VII	$T = T_0$ at $\varphi = 0, 2\pi\alpha$ and $\eta = \eta_0, \eta_1$	$1 \leqslant \xi \leqslant \xi_0$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi) L^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta) \sin(\frac{m}{2\alpha}\varphi)$	truncated
	$T = T_0 + f_0(\eta, \varphi)$ at $\xi = \xi_0$	$\eta_1 \leqslant \eta \leqslant \eta_0$	λ_{mn} from (26)	sector spheroid
VIII	$T = T_0$ at $\varphi = 0, 2\pi\alpha$ and $\xi = \xi_0$	$1 \leqslant \xi \leqslant \xi_0$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{m=1}^{\infty} [A_{mn} P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta) + B_{mn} Q^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\eta)] P^{\frac{m}{2\alpha}}_{\lambda_{mn}}(\xi) \sin(\frac{m}{2\alpha}\varphi)$	truncated
	$T = T_0 + g_i(\xi, \varphi)$ at $\eta = \eta_i\ (i = 0, 1)$	$\eta_1 \leqslant \eta \leqslant \eta_0$ $0 \leqslant \varphi \leqslant 2\pi\alpha$	λ_{mn} from $P^{\frac{m}{2\alpha}}_{\lambda}(\xi_0) = 0$	sector spheroid
IX	$T = T_0$ at $\eta = \eta_0, \eta_1$	$\xi_1 \leqslant \xi \leqslant \xi_0$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [A_{mn} P^{m}_{\lambda_{mn}}(\xi) + B_{mn} Q^{m}_{\lambda_{mn}}(\xi)] L^{m}_{\lambda_{mn}}(\eta) \cos m\varphi$	truncated
	$T = T_0 + f_i(\eta, \varphi)$ at $\xi = \xi_i\ (i = 0, 1)$	$\eta_1 \leqslant \eta \leqslant \eta_0$ $0 \leqslant \varphi \leqslant 2\pi$	λ_{mn} from (26) for $\frac{m}{2\alpha} \to m$	annular spheroid
X	$T = T_0$ at $\xi = \xi_0, \xi_1$	$\xi_1 \leqslant \xi \leqslant \xi_0$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [A_{mn} P^{m}_{\lambda_{mn}}(\eta) + B_{m} Q^{m}_{\lambda_{mn}}(\eta)] L^{m}_{\lambda_{mn}}(\xi) \cos m\varphi$	truncated
	$T = T_0 + g_i(\xi, \varphi)$ at $\eta = \eta_i\ (i = 0, 1)$	$\eta_1 \leqslant \eta \leqslant \eta_0$ $0 \leqslant \varphi < 2\pi$	λ_{mn} from (7) for $\frac{m}{2\alpha} \to m$	annular spheroid
XI	$T = T_0$ at $\eta = \eta_0 \cdot \eta_1$	$1 \leqslant \xi \leqslant \xi_0$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} P^{m}_{\lambda_{mn}}(\xi) L^{m}_{\lambda_{mn}}(\eta) \cos m\varphi$	truncated
	$T = T_0 + f_0(\eta, \varphi)$ at $\xi = \xi_0$	$\eta_1 \leqslant \eta \leqslant \eta_0$ $0 \leqslant \varphi < 2\pi$	λ_{mn} from (26) for $\frac{m}{2\alpha} \to m$	spheroid
XII	$T = T_0$ at $\xi = \xi_0$	$1 \leqslant \xi \leqslant \xi_0$	$T(\xi, \eta, \varphi) = T_0 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [A_{mn} P^{m}_{\lambda_{mn}(\eta)} + B_{m} Q^{m}_{\lambda_{mn}}(\eta)] P^{m}_{\lambda_{mn}}(\xi) \cos m\varphi$	truncated
	$T = T_0 + g_i(\xi, \varphi)$ at $\eta = \eta_i\ (i = 0, 1)$	$\eta_1 \leqslant \eta \leqslant \eta_0$ $0 \leqslant \varphi < 2\pi$	λ_{mn} from $P^{m}_{\lambda}(\xi_0) = 0$	spheroid

Special cases

In the following, we shall evaluate a few special cases numerically and determine the local temperature distribution and isothermal lines.

1. Prolate spheroid

First of all we shall treat a complete prolate spheroid for which the temperature distribution at the surface $\xi = \xi_0 = 1.5$ is given by

$$T(\eta) = T_0 + T_1 \sin \pi\eta$$

The solution as obtained from the above results yields the expression ($m=0$)

$$T(\xi, \eta) = T_0 + \sum_{n=0}^{\infty} A_{2n+1} P^0_{2n+1}(\xi) P^0_{2n+1}(\eta)$$

as obtained from Eq. (5). With the orthogonality relation, the integration constant are given by

$$A_{2n+1} = \frac{(4n+3)T_1}{2P^0_{2n+1}(\xi_0)} \int_{-1}^{+1} \sin \pi\eta \, P^0_{2n+1}(\eta)\, d\eta$$

which may be integrated numerically or analytically in the following way. With tl series representation of $\sin \pi\eta$ and the representation of the odd powers of η [5

$$\int_{-1}^{+1} \sin \pi\eta \, P^0_{2n+1}(\eta)\, d\eta = \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m+1}}{(2m+1)!(2m+3)} \left\{ 3 \int_{-1}^{+1} P^0_1(\eta) P^0_{2n+1}(\eta) + \right.$$

$$+7 \frac{2m}{(2m+5)} \int_{-1}^{+1} P^0_{2n+1}(\eta)\, d\eta + \frac{11(2m-2)2m}{(2m+5)(2m+7)} \int_{-1}^{+1} P^0_5(\eta) P^0_{2n+1}(\eta)\, d\eta + \ldots$$

$$+ (4m+3) \frac{2m(2m-2)\ldots 2}{(2m+5)(2m+7)\ldots(4m+3)} \int_{-1}^{+1} P^0_{2m+1}(\eta) P^0_{2n+1}(\eta)\, d\eta$$

A faster procedure is found in [11], in which the integral is given as

$$\int_{-1}^{+1} \sin(\pi\eta) P^0_{2n+1}(x)\, dx = \sqrt{2}\,(-1)^n J_{2n+3/2}(\pi)$$

(note of the reviewer)

From this, the magnitude of A_{2n+1} ($n = 0, 1, 2, \ldots$) may be found as

$$A_{2n+1} = \frac{(4n+3)T_1(-1)^n}{\sqrt{2}\,P^0_{2n+1}(\xi_0)} \cdot J_{2n+3/2}(\pi)$$

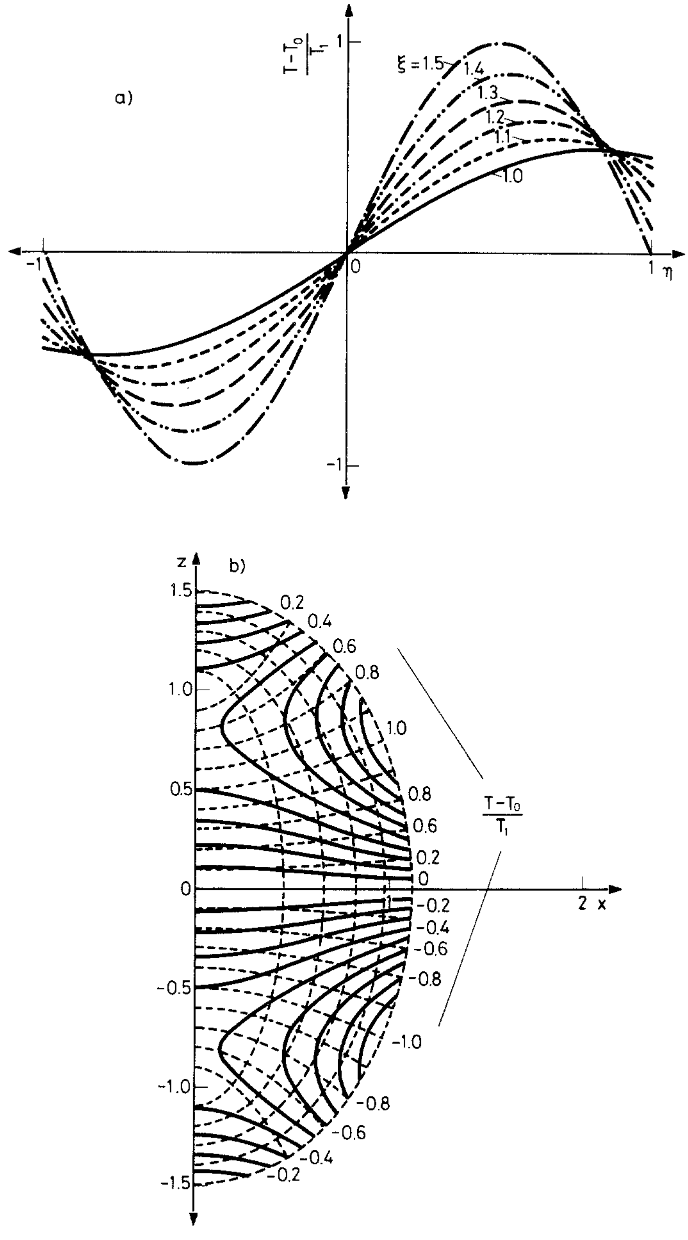


Fig. 3 Temperature distribution and isothermal lines in a prolate spheroid

The temperature distribution and the isothermal lines are presented in Figs 3a and 3b as functions of η with ξ as a parameter.

2. Truncated sectorial prolate spheroid

In the second numerical evaluation of the above results, we treat a truncated sectorial prolate spheroid, which at $\varphi = 0, \frac{\pi}{2}$ and $\eta = 0$ exhibits a temperature $T = T_0$. At the spheroidal surface $\xi = \xi_0 = 1.5$, the applied temperature distribution is given by

$$T = T_0 + \frac{64}{\pi^2} T_1 \left(\varphi - \frac{\pi}{2}\right) \varphi\eta(1-\eta) \quad \text{at} \quad \xi = \xi_0$$

The solution (case V in Table 1) gives the expression

$$T(\xi, \eta, \varphi) = T_0 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} P^{2m}_{\lambda_{mn}}(\xi) P^{2m}_{\lambda_{mn}}(\eta) \sin 2m\varphi$$

where the λ_{mn} are obtained from $P^{2m}_{\lambda}(o) = 0$. The roots of $P^{m/2\alpha}_{\lambda}(\eta_0) = 0$ may be found in [7] and for $m/2$ (m integer) in [8]. In this particular case, the λ_{mn} are integers; for odd integers m they are even integers, and for even integers m they are odd integers (see [7, 8]). The numerical results are presented in Figs 4a–d.

3. Truncated sectorial prolate spheroid

For a truncated sectorial prolate spheroid, which at $\varphi = 0, \frac{\pi}{2}$ and at $\xi = \xi_0 = 1.5$ exhibits a temperature $T = T_0$ and at the surface $\eta = \eta_0 = 0$ the given temperature distribution

$$T(\xi, \eta) = T_0 + T_1 \varphi \left(\varphi - \frac{\pi}{2}\right)(\xi - 1)\left(\xi - \frac{3}{2}\right)$$

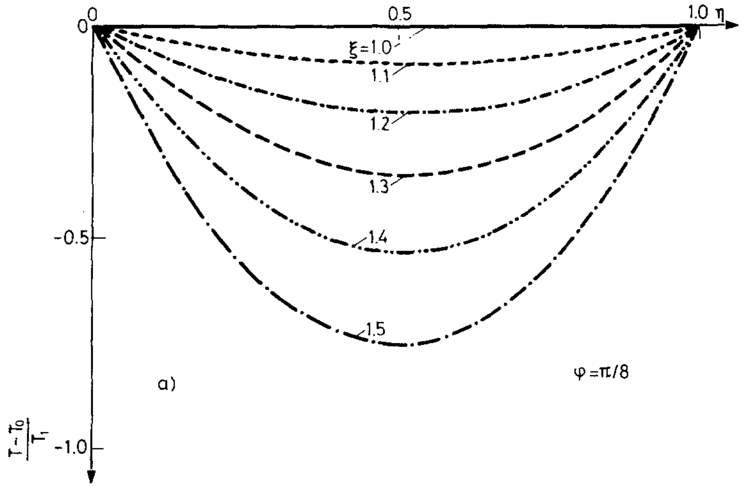
the solution (case VI in Table 1) yields the expression

$$T(\xi, \eta, \varphi) = T_0 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{2m-1n} P^{2(2m-1)}_{\lambda_{2m-1n}}(\xi) P^{2(2m-1)}_{\lambda_{2m-1\mu}}(\eta) \sin [2(2\text{m}-1)\varphi]$$

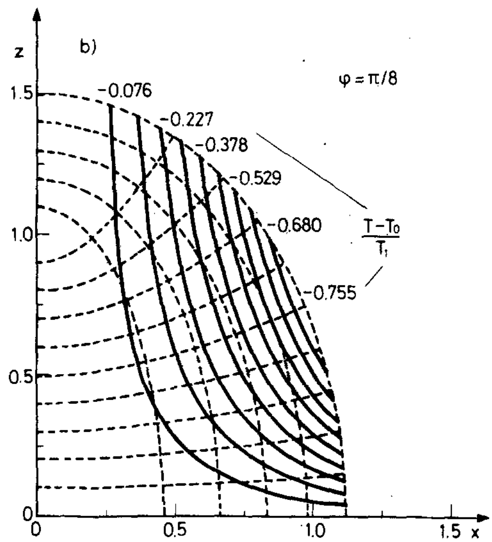
where the roots λ_{2m-1n} are obtained from $P^{4m-2}_{\lambda}(1.5) = 0$.

The roots of $P^{m}_{\lambda}(\xi_0) = 0$ ($\xi_0 > 1$) may be found in Table 2, which represents the imaginary part of the roots $\lambda_{mn} = -\frac{1}{2} + i\alpha^*_{mn}$, i.e. α^*_{mn}.

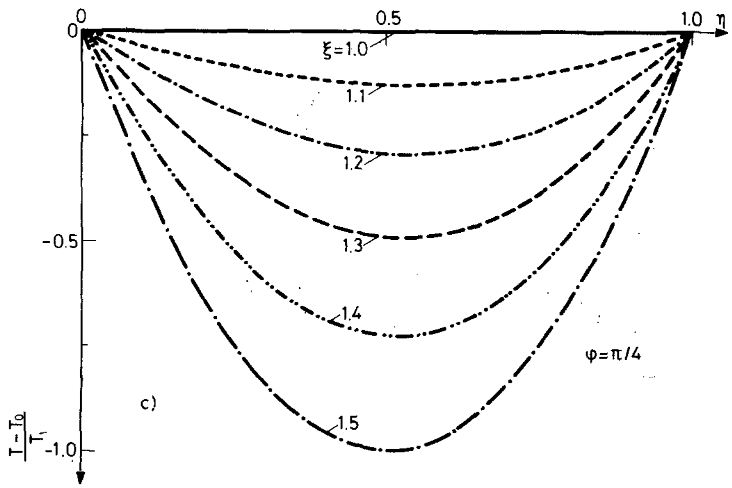
In order to satisfy the boundary condition on the surface $\eta = \eta_0 = 0$ and to determine the magnitude of the integration constants B_{2m-1n}, we must expand $\left(0 \leqslant \varphi \leqslant \frac{\pi}{2}, 1 \leqslant \xi \leqslant 1.5 \text{ and } 0 \leqslant \eta \leqslant 1\right)$



0
0.5
1.0 η
ξ=1.0
1.1
1.2
1.3
1.4
1.5
-0.5
-1.0
(T−T0)/T1
a)
φ=π/8



z
b)
φ=π/8
1.5
1.0
0.5
0
-0.076
-0.227
-0.378
-0.529
-0.680
-0.755
(T−T0)/T1
0.5
1.0
1.5 x



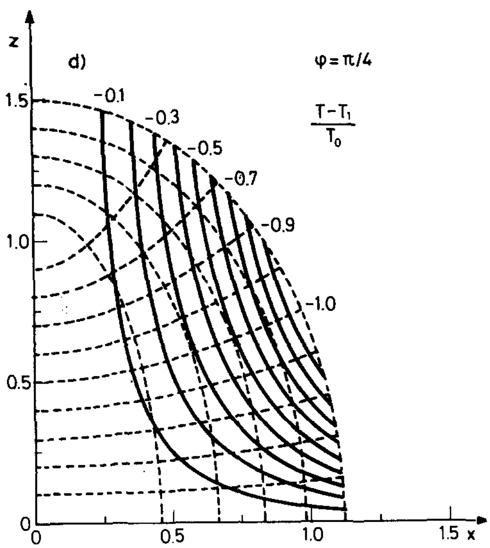


Fig. 4 Temperature distribution and isothermal lines in φ-planes of a truncated sectorial prolate spheroid with an angular width of $\frac{\pi}{2}\left(\alpha = \frac{1}{4}\right)$; $T(\xi, \eta)$ on $\xi = \xi_0 = 1.5$, T_0 on $\varphi = 0, \frac{\pi}{2}$ and $\eta = \eta_0 = 0$

$$T_1\varphi\left(\varphi-\frac{\pi}{2}\right)(\xi-1)\left(\xi-\frac{3}{2}\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\gamma_{2m-1n}P'^{2(2m-1)}_{\lambda_{2m-1n}}(\xi)\sin(4m-2)\varphi$$

into a Legendre–Fourier-sine series. It is

$$T_1\varphi\left(\varphi-\frac{\pi}{2}\right)=-\frac{2T_1}{\pi}\sum_{m=1}^{\infty}\frac{\sin(4m-2)\varphi}{(2m-1)^3}$$

while $\left(\xi_0=\frac{3}{2}\right)$

$$(\xi-1)(\xi-\xi_0)=\sum_{n=1}^{\infty}\frac{2\lambda_{2m-1n}\cdot\int_1^{\xi_0}(\xi-1)(\xi-\xi_0)P^{2(2m-1)}_{\lambda_{2m-1n}}(\xi)\,d\xi\,.}{(1-\xi_0^2)\dfrac{\partial P^{2(2m-1)}_{\lambda_{2m-1n}}(\xi_0)}{\partial\xi}\cdot\dfrac{\partial P^{2(2m-1)}_{\lambda_{2m-1n}}(\xi_0)}{\partial\lambda_{2m-1n}}}$$

Thus, the integration constants are given by

$$B_{2m-1n}=\frac{8T_1(2\lambda_{2m-1n}+1)}{5\pi(2m-1)^3\dfrac{\partial P^{4m-2}_{\lambda_{2m-1n}}(1.5)}{\partial\xi}\cdot\dfrac{\partial P^{4m-2}_{\lambda_{2m-1n}}(1.5)}{\partial\lambda_{2m-1n}}\cdot P^{4m-2}_{\lambda_{2m-1n}}(O)}\,.$$

$$\cdot\int_1^{1.5}(\xi-1)\left(\xi-\frac{3}{2}\right)P^{4m-2}_{\lambda_{2m-1n}}(\xi)\,d\xi$$

The introduction of these values into the above expression yields the temperature distribution for this particular case. The results for $(T-T_0)/T_1$ are presented in Figs 5a, 5b and 5c for the sectorial planes $\varphi=\pi/16, \pi/8$ and $\pi/4$. Symmetry appears at $\varphi=\pi/4$.

Numerical evaluations

For some special cases, numerical evaluations have been performed, which exhibit the temperature distribution on different prolate spheroids, in various azimuthal planes $\varphi=$ const. and the isothermal lines. For a complete simple prolate spheroid with given temperature distribution at a spheroidal surface, in this case a distribution $T(\eta)=T_0+T_1\sin\pi\eta$ at $\xi_0=1.5$, the results are presented in Figs 3a and 3b. On the various prolate spheroids $\xi\leqslant\xi_0=1.5$, the temperature distribution $(T-T_0)/T_1$ is shown as a function of the coordinate η in Fig. 3a. It may be noted that the maximum temperature, which appears on the surface at $\eta=0.5$, shifts towards the top and bottom of the spheroid as we move to inner prolate

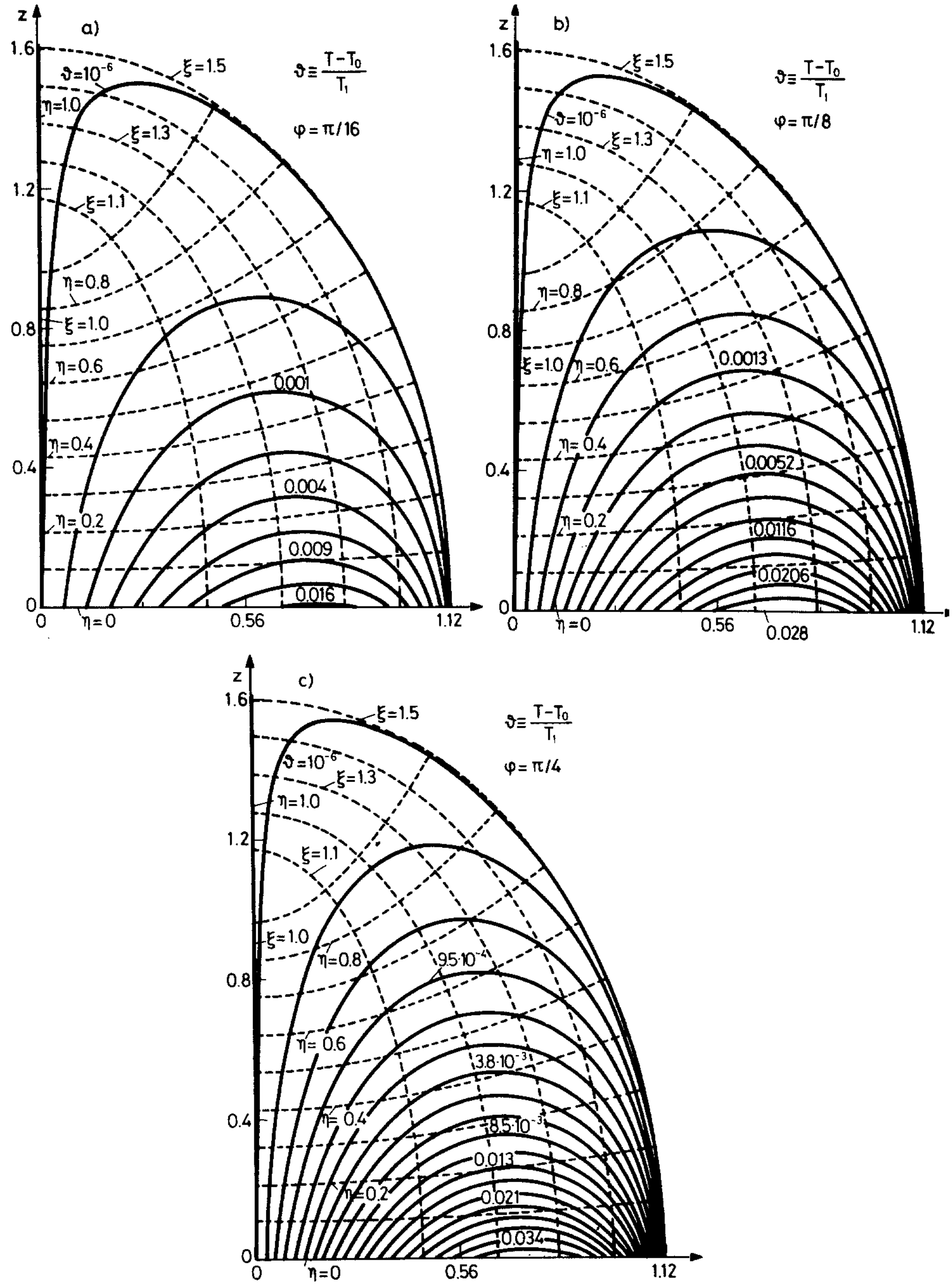


Fig. 5 Temperature distribution and isothermal lines in φ-planes of a truncated sectorial prolate spheroid with an angular width of $\frac{\pi}{2}\left(\alpha = \frac{1}{4}\right)$; $T(\xi, \eta)$ on $\eta = \eta_0 = 0$, T_0 on $\varphi = 0, \frac{\pi}{2}$ and $\xi = \xi_0 = 1.5$

spheroids $\xi < 1.5$ and decreases in addition. The isothermal lines are presented as solid lines in Fig. 3b, where the dotted lines represent the coordinate surfaces ξ = const. and η = const. respectively. In the equator plane the temperature is T_0 and is also exhibited at the poles. The temperature $(T-T_0)/T_1 = \pm 0.5$ does not penetrate through the whole spheroid. A more complicated geometry has been chosen in the following two problems, where a truncated sectorial prolate spheroid has been treated (see sections 3.2 and 3.3).

This is a quarter spheroid which is bounded by azimuthal planes at $\varphi = 0$ and $\pi/2$ and the equator plane, at which a constant temperature T_0 was chosen. At the spheroidal surface $\xi_0 = 1.5$, the given temperature distribution is $T = T_0 + (64T_1/\pi^2)\varphi\left(\varphi - \frac{\pi}{2}\right)\eta(1-\eta)$. The temperature distribution inside this quarter spheroid is presented for various azimuthal planes $\varphi = \frac{\pi}{8}$ (Figs 4a and 4b) and $\varphi = \frac{\pi}{4}$ (Figs 4c and 4d). Because of symmetry to $\varphi = \frac{\pi}{4}$, the plane $\varphi = \frac{3\pi}{8}$ has not been presented separately, since it yields the results of the temperature distribution in the azimuthal plane $\varphi = \frac{\pi}{8}$. Figure 4a presents the temperature $(T-T_0)/T_1$ on the spheroidal surface lines ξ = const. < 1.5 as a function of the coordinate η. The temperature decreases with decreasing ξ, it increases from the equator towards $\eta = 0.5$ to a maximum, and it then decreases again. The isothermal lines in the azimuthal plane $\varphi = \frac{\pi}{8}$ are presented in Fig. 4b and show decreasing magnitude towards the corner of the quarter spheroid. Similar results may be found for the plane $\varphi = \frac{\pi}{4}$, as is exhibited in Figs 4c and 4d, where a less rapid decrease in the magnitude of the temperature may be observed. In these investigations, the roots of the associated Legendre function with respect to the degree for an argument smaller than unity, i.e. $P^{2m}_{\lambda_{mn}}(\eta_0) = 0$, must be employed [7]. In this particular case for $\eta_0 = 0$ they are particularly simple, i.e. they are integers, but would present no problem at all for other configurations or for $\eta_0 \neq 0$ (see [7, 8]). A more complicated case is the third case treated, when for a quarter spheroid the azimuthal planes $\varphi = 0$ and $\frac{\pi}{2}$ are kept at temperature T_0, where the spheroidal surface $\xi_0 = 1.5$ is kept at $T = T_0$ and where at $\eta_0 = 0$ the temperature distribution $T(\xi, \eta) = T_0 + T_1\varphi\left(\varphi - \frac{\pi}{2}\right)\cdot(\xi - 1)\left(\xi - \frac{3}{2}\right)$ is enforced. In this case the roots of the associated Legendre function $P^{4m-2}_{\lambda_{mn}}(\xi_0) = 0$ with respect to the degree λ for an

Table 2 Root parts α^*_{mn} of roots of $P^m_\lambda(\xi) = 0 \quad \lambda_{mn} = -\frac{1}{2} + i\alpha^*_{mn}$

$\xi_0 = 1.1$

n/m	0	1	2	3	4	5
1	5.429161	8.624073	11.524727	14.283808	16.956396	19.569522
2	12.448017	15.808447	18.943706	21.940315	24.841014	27.670536
3	19.511458	22.930139	26.172739	29.293006	32.322170	35.280237
4	26.584907	30.033416	33.338158	36.535555	39.649000	42.694558
5	33.662147	37.128798	40.474179	43.724556	46.897995	50.007541
6	40.741220	44.220074	47.594139	50.883238	54.101705	57.260294'
7	47.821317	51.308942	54.704345	58.023015	61.276583	64.474045
8	54.902044	58.396277	61.808181	65.150185	68.431912	71.660985
9	61.983186	65.482578	68.907622	72.268517	75.573347	78.828653
10	69.064615	72.568146	76.003905	79.380408	82.704545	85.981981

n/m	6	7	8	9	10	11
1	22.138738	24.673926	27.181809	29.667198	32.133666	34.583954
2	30.444671	33.174207	35.866889	38.528492	41.163455	43.775272
3	38.180918	41.034078	43.847092	46.625637	49.374194	52.096366
4	45.683811	48.625431	51.526104	54.391111	57.224710	60.030387
5	53.062974	56.071844	59.040120	61.972619	64.873288	67.745413
6	60.367316	63.429349	66.451702	69.438734	72.394071	75.320771
7	67.622527	70.727782	73.794538	76.826731	79.827681	82.800217
8	74.843568	77.984725	81.088674	84.158976	87.198667	90.210359
9	82.039810	85.211298	88.346902	91.449847	94.522910	97.568504
10	89.217443	92.414920	95.577817	98.709066	101.811214	104.886488

$\xi_0 = 1.2$

n/m	0	1	2	3	4	5
1	3.874618	6.136886	8.178078	10.113289	11.983893	13.810156
2	8-874141	11.261691	13.479630	15.593290	17.634920	19.623161
3	13.907566	16.339073	18.637904	20.844631	22.982827	25.067590
4	18.948554	21.402556	23.748218	26.013029	28.214645	30.365190
5	23.992403	26.460083	28.836374	31.141092	33.387874	35.586565
6	29.037635	31.514514	33.912422	36.246313	38.527004	40.762612
7	34.083639	36.567129	38.981119	41.337291	43.644420	45.909298
8	39.130117	41.618589	44.045014	46.418768	48.747110	51.035794
9	44.176909	46.669269	49.105598	51.493586	53.839334	56.147776
10	49.203917	51.719397	54.163801	59.029347	58.923845	61.248957

n/m	6	7	8	9	10	11
1	15.603719	17.371968	19.119923	20.851172	22.568380	24.273587
2	21.569905	23.483268	25.369071	27.231653	29.074344	30.899765
3	27.109259	29.115271	31.091181	33.041266	34.968895	36.876779
4	32.473405	34.545831	36.587510	38.602425	40.593781	42.564205
5	37.744546	39.867515	41.959975	44.025558	46.067242	48.087504
6	42.959409	45.122359	47.255470	49.362028	51.444771	53.506003
7	48.137300	50.332770	52.499276	54.639788	56.756810	58.852478
8	53.298471	55.511961	57.706451	59.875626	62.021778	64.146876
9	58.422969	60.668294	62.886606	65.080339	67.251595	69.402182
10	63.542456	65.807350	68.046203	70.261214	72.454592	74.627569

$\xi_0 = 1.3$

n/m	0	1	2	3	4	5
1	3.191930	5.041666	6.700756	8.268773	9.781393	11.256052
2	7.303025	9.261568	11.073464	12.795339	14.455122	16.068940
3	11.443692	13.440307	15.322168	17.124435	18.867531	20.564533
4	15.590919	17.607004	19.529376	21.381848	23.179717	24.933488
5	19.740623	21.768546	23.717413	25.0604389	27.441287	29.236638
6	23.891525	25.927407	27.894975	29.807179	31.673388	33.500635
7	28.043095	30.084698	32.066178	33.997650	35.886718	37.739261
8	32.195076	34.240989	36.233226	38.179906	40.087315	41.960437
9	36.347328	38.396604	40.397409	42.356401	44.278862	46.169066
10	40.499767	42.551742	44.559533	46.528698	48.463739	50.368357

n/m	6	7	8	9	10	11
1	12.702751	14.127828	15.535586	16.929097	18.310641	19.681963
2	17.647071	19.196523	20.722319	22.228198	23.717028	25.191067
3	22.224390	23.853537	25.456779	27.037810	28.599542	30.144313
4	26.650730	28.337109	29.996988	31.633813	33.250356	34.848889
5	30.996835	32.726813	34.430473	36.110959	37.770848	39.412283
6	35.294353	37.058844	38.797578	40.513402	42.208685	43.885424
7	39.559938	41.352514	43.120076	44.865200	46.590053	48.296485
8	43.803301	45.619218	47.410948	49.180817	50.930810	52.662634
9	48.030527	49.866175	51.678484	53.469563	55.241228	56.995055
10	52.245641	54.098201	55.928262	57.737741	59.528303	61.301405

(Table 2. cont.)

$\xi_0 = 1.4$

n/m	0	1	2	3	4	5
1	2.788197	4.392405	5.823108	7.171212	8.469178	9.732840
2	6.373017	8.076923	9.647018	11.135101	12.566696	13.956527
3	9.985045	11.723747	13.357704	14.919068	16.426515	17.891999
4	13.603077	15.359564	17.030502	18.637662	20.195031	21.712216
5	17.223373	18.990671	20.685979	22.324441	23.917402	25.472483
6	20.844763	22.619330	24.331529	25.993196	27.612896	29.197060
7	24.466764	26.246554	27.971454	29.650708	31.291264	32.898501
8	28.089140	29.872865	31.607587	33.300721	34.958015	36.584026
9	31.711763	33.498560	35.241104	36.945477	38.616508	40.258104
10	35.334557	37.123818	38.872740	40.586400	42.268913	43.923661

n/m	6	7	8	9	10	11
1	10.971266	12.190194	13.393518	14.584014	15.763742	16.934275
2	15.313968	16.645406	17.955413	19.247389	20.523944	21.787126
3	19.323694	20.727478	22.107733	23.467829	24.810418	26.137633
4	23.196134	24.651964	26.083690	27.494459	28.886802	30.262789
5	26.995528	28.491048	29.962608	31.413081	32.844821	34.259784
6	30.750655	32.277616	33.781112	35.263748	36.727687	38.174754
7	34.476677	36. 029232	37.558993	39.068311	40.559170	42.033257
8	38.182435	39.756282	41.308058	42.839921	44.353676	45.850883
9	41.873477	43.465306	45.035854	46.587047	48.120548	49.637796
10	45.553470	47.160724	48.747459	50.315427	51.866151	53.400962

$\xi_0 = 1.5$

n/m	0	1	2	3	4	5
1	2.514703	3.951624	5.226155	6.423639	7.574451	8.693376
2	5.742473	7.273313	8.678589	10.007028	11.282617	12.519181
3	8.995975	10.559518	12.024735	13.421866	14.768473	16.075792
4	12.255127	13.835377	15.335311	16.775400	18.168799	19.524539
5	15.516407	17.106810	18.629453	20.099086	21.525864	22.917122
6	18.778715	20.375944	21.914617	23.405858	24.857720	26.276249
7	22.041596	23.643731	25.194329	26.702077	28.173509	29.613680
8	25.304830	26.910660	28.470479	29.991258	31.478398	32.936182
9	28.568296	30.177010	31.744174	33.275503	34.775536	36.247944
10	31.831922	33.442951	35.016103	36.556151	38.066960	39.551712

(Table 2. cont.)

n/m	6	7	8	9	10	11
1	9.788870	10.866277	11.929221	12.980281	14.021370	15.053951
2	13.725514	14.907598	16.069714	17.215043	18.346023	19.464568
3	17.351504	18.601127	19.828772	21.037601	22.230099	23.408262
4	20.849124	22.147419	23.423169	24.679327	25.918268	27.141936
5	24.278357	25.613810	26.926831	28.220119	29.495886	30.755967
6	27.666118	20.031025	30.373950	31.697341	33.003232	34.293336
7	31.026596	32.415492	33.783024	35.131405	36.462496	37.777887
8	34.368075	35.776923	37.165099	38.534602	39.887139	41.224177
9	37.695750	39.121472	40.527237	41.914858	43.285897	44.641707
10	41.013061	42.453252	43.874196	45.277543	46.664724	48.036991

$\xi_0 = 1.6$

n/m	0	1	2	3	4	5
1	2.314202	3.627838	4.786920	5.872914	6.914721	7.926382
2	5.279847	6.683428	7.967196	9.177759	10.338042	11.461245
3	8.270228	9.705061	11.046036	12.322074	13.549969	14.740460
4	11.266013	12.716805	14.090922	15.407936	16.680427	17.917031
5	14.263833	15.724322	17.120119	18.465323	19.769646	21.040101
6	17.262635	18.729643	20.140743	21.506570	22.834822	24.131280
7	20.261985	21.733677	23.156163	24.537746	25.884671	27.201772
8	23.261671	24.736892	26.168183	27.562203	28.924118	30.258022
9	26.261579	27.739554	29.177857	30.581950	31.956169	33.304028
10	29.261641	30.741826	32.185846	33.598262	34.982767	35.342404

n/m	6	7	8	9	10	11
1	8.915915	9.888383	10.847211	11.794841	12.733080	13.663308
2	12.555752	13.627263	14.679855	15.716552	16.739670	17.751029
3	15.900885	17.036512	18.151265	19.248157	20.329557	21.397366
4	19.123965	20.305879	21.466348	22.608192	23.733679	24.844661
5	22.281944	23.499231	24.695161	25.872315	27.032803	28.178379
6	25.400414	26.645758	27.870163	29.075967	30.265114	31.439240
7	28.492879	29.761087	31.008934	32.238535	33.451672	34.649861
8	31.567228	32.854458	34.121980	35.371707	36.605269	37.824068
9	34.628413	35.931734	37.216025	38.483019	39.734208	40.970889
10	37.679711	38.996832	40.295598	41.577584	42.844159	44.096519

Table 3 Root parts β^*_{mn} of root of $\frac{\partial P^m_{\lambda'}}{\partial \xi}(\xi_0) = 0 \quad \lambda'_{mn} = -\frac{1}{2} + i\beta^*_{mn}$

$\xi_0 = 1.1$

n/m	0	1	2	3	4	5
1	8.624073	4.063996	6.753910	9.284164	11.741139	14.154450
2	15.808447	11.994973	15.065856	17.979722	20.793352	23.536126
3	22.930139	19.229568	22.440536	25.515587	28.493710	31.398373
4	30.033416	26.379581	29.665420	32.833995	35.913355	38.921995
5	37.128798	33.500520	36.833310	40.063633	43.212552	46.294849
6	44.220074	40.607906	43.972974	47.247290	50.447223	53.584797
7	51.308942	47.707855	51.096508	54.403817	57.642863	60.823593
8	58.396277	54.803278	58.209928	61.542925	64.812955	68.028375
9	65.482578	61.895741	65.316580	68.670156	71.965360	75.209304
10	72.568146	68.986160	72.418477	75.788914	79.104986	82.372816

n/m	6	7	8	9	10	11
1	16.538226	18.900441	21.246095	23.578553	25.900203	28.212806
2	26.225549	28.873023	31.486453	34.071582	36.632730	39.173240
3	34.244984	37.044291	39.804136	42.530444	45.227821	47.899923
4	41.872864	44.775436	47.636889	50.462809	53.257647	56.025008
5	49.321363	52.300319	55.238130	58.139907	61.009804	63.851246
6	55.669150	59.707413	62.705279	65.667374	68.597519	71.498912
7	63.953781	67.039639	70.086217	73.097690	76.077554	79.028770
8	71.195865	74.320862	77.407864	80.460637	83.482375	86.475812
9	78.407782	81.565587	84.686735	87.774635	90.832204	93.861963
10	85.597471	88.783201	91.933613	95.051795	98.140417	101.201807

$\xi_0 = 1.2$

n/m	0	1	2	3	4	5
1	6.136886	2.836772	4.724600	6.491296	8.202850	9.881681
2	11.261691	8.532524	10.702181	12.753829	14.730281	16.653657
3	16.339073	13.695077	15.970136	18.142894	20.242776	22.287482
4	21.402556	18.793794	21.125239	23.368375	25.554372	27.667186
5	26.460083	23.870585	26.237400	28.527030	30.755370	32.933576
6	31.514514	28.937157	31.328294	33.651058	35.917804	38.137605
7	36.567129	33.998124	36.407032	38.7546652	41.050851	43.303138
8	41.618589	39.055680	41.478150	43.845122	46.164659	48.443066
9	46.669269	44.111005	46.544167	48.926644	51.265144	53.565032
10	51.719397	49.164789	51.606600	54.001782	56.356003	60.821218

(Table 3. cont.)

n/m	6	7	8	9	10	11
1	9.301440	10.618417	11.924836	13.222863	14.514029	15.799461
2	15.131930	16.628906	18.104106	19.561231	21.003065	22.431756
3	19.891726	21.486198	23.055254	24.602742	26.131649	27.544344
4	24.398906	26.060053	27.694619	29.306243	30.897823	32.471705
5	28.789954	30.500561	32.184511	33.845141	35.485158	37.106796
6	33.115671	34.864747	36.587613	38.287292	39.966278	41.626657
7	37.400370	39.180353	40.934826	42.666518	44.377712	46.070343
8	41.657411	43.462823	45.243559	47.002054	48.740406	50.460399
9	45.894833	47.721587	49.524506	51.305816	53.067425	54.810982
10	50.117791	51.962713	53.784659	55.585649	57.367429	59.131520

$\xi_0 = 1.3$

n/m	0	1	2	3	4	5
1	5.041666	2.286480	3.817547	5.243078	6.620961	7.970678
2	9.261568	7.007343	8.777651	10.446165	12.049931	13.608066
3	13.440307	11.259835	13.121181	14.894150	16.604226	18.266735
4	17.607004	15.457023	17.367117	19.200876	20.976639	22.706496
5	21.768546	19.635233	21.575914	23.449857	25.270832	27.048502
6	25.927407	23.804599	25.766315	27.668905	29.523067	31.336665
7	30.084698	27.969115	29.946202	31.870284	33.749906	35.591574
8	34.240989	32.130680	34.119497	36.060315	37.960114	39.824358
9	38.396604	36.290313	38.288381	40.242608	42.158804	44.041616
10	42.551742	40.448615	42.454164	44.419380	46.349170	48.247537

n/m	6	7	8	9	10	11
1	9.301440	10.618417	11.924836	13.222863	14.514029	15.799461
2	15.131930	16.628906	18.104106	19.561231	21.003065	22.431756
3	19.891726	21.486198	23.055254	24.602742	26.131649	27.644344
4	24.398906	26.060053	27.694619	29.306243	30.897823	32.471705
5	28.789954	30.500561	32.184511	33.845141	35.485158	37.106796
6	33.115671	34.864747	36.587613	38.287292	39.966278	41.626657
7	37.400370	39.180353	40.934826	42.666518	44.377712	47.070343
8	41.657411	43.462832	45.243559	47.002054	48.740406	50.460399
9	45.894833	47.721587	49.524506	51.305816	53.067425	54.810982
10	50.117791	51.962713	53.784659	55.585649	57.367429	59.131520

(Table 3. cont.)

$\xi_0 = 1.4$

n/m	0	1	2	3	4	5
1	4.392405	1.954688	3.272694	4.493616	5.671068	6.822922
2	8.076923	6.102925	7.635196	9.074807	10.455585	11.794960
3	11.723747	9.817152	11.432417	12.967106	14.444517	15.878652
4	15.359564	13.480818	15.140561	16.730667	18.267904	19.763323
5	18.990671	17.127145	18.814805	20.441565	22.020015	23.558991
6	22.619330	20.765396	22.472257	24.125155	25.733872	27.305596
7	26.246554	24.399218	26.120110	27.792621	29.424565	31.021886
8	29.872865	28.030344	29.761945	31.449732	33.100085	34.717997
9	33.498560	31.659707	33.399753	35.099779	36.765094	38.399948
10	37.123818	35.287854	37.034730	38.744786	40.422509	42.071555

n/m	6	7	8	9	10	11
1	7.957592	9.079787	10.192441	11.297517	12.396405	13.490126
2	13.103279	14.387258	15.651540	16.899489	18.133627	19.355904
3	17.278685	18.651005	20.000261	21.329960	22.642822	23.941003
4	21.224654	22.657548	24.066278	25.454167	26.823856	28.177484
5	25.064969	26.542861	27.996495	29.428917	30.842601	32.239584
6	28.845789	30.358712	31.847765	33.315711	34.764828	36.197022
7	32.589232	34.130316	35.648155	37.145245	38.623672	40.085204
8	36.307463	37.871733	39.413494	40.934998	42.438153	43.924591
9	40.007803	41.591525	43.153514	44.695807	46.220148	47.728043
10	43.694952	45.295242	46.874578	48.434809	49.977531	51.504133

$\xi_0 = 1.5$

n/m	0	1	2	3	4	5
1	3.951624	1.725767	2.898365	3.979077	5.018985	6.034959
2	7.273313	5.488747	6.858632	8.141779	9.369951	10.559492
3	10.559518	8.838304	10.286002	11.658152	12.976656	14.254671
4	13.835377	12.140320	13.629757	15.053861	16.428400	17.763765
5	17.106810	15.426048	16.941687	18.400176	19.813351	21.189521
6	20.375944	18.704190	20.237854	21.720869	23.162431	24.569301
7	23.643731	21.978171	23.525009	25.026431	26.489784	27.920658
8	26.910660	25.249622	26.806513	28.322273	29.802901	31.253091
9	30.177010	28.519417	30.084236	31.611486	33.106155	34.572240
10	33.442951	31.788070	33.359302	34.895967	36.402282	37.881683

(Table 3. cont.)

n/*m*	6	7	8	9	10	11
1	7.034903	8.023230	9.002683	9.975097	10.941765	11.903640
2	11.720088	12.858023	13.977637	15.082067	16.173668	17.254257
3	15.500812	16.721070	17.919805	19.100304	20.265112	21.416246
4	19.067220	20.344073	21.598337	22.833129	24.050928	25.253739
5	22.534775	23.853731	25.149983	26.426395	27.685288	28.928574
6	25.946612	27.298372	28.627777	29.937422	31.229449	32.505647
7	29.323423	30.701569	32.057936	33.394866	34.714323	36.017965
8	32.676598	34.076479	35.455261	36.815060	38.157671	39.484627
9	36.012995	37.431115	38.828858	40.208138	41.570596	42.917647
10	39.337019	40.770678	42.184686	43.580781	44.960463	46.325039

$\xi_0 = 1.6$

n/*m*	0	1	2	3	4	5
1	3.627838	1.554968	2.620389	3.597336	4.535320	5.450537
2	6.683428	5.037472	6.287554	7.455075	8.570360	9.648989
3	9.705061	8.119654	9.443928	10.696173	11.897327	13.059961
4	12.716805	11.156384	12.520487	13.822262	15.076787	16.293994
5	15.724322	14.177552	15.566666	16.901241	18.192597	19.448677
6	18.729643	17.191474	18.597796	19.955774	21.274204	22.559548
7	21.733677	20.201424	21.620321	22.995868	24.335086	25.643321
8	24.736892	23.208957	24.637452	26.026684	27.382386	28.709046
9	27.739554	26.214908	27.650973	29.051172	30.420274	31.762100
10	30.741826	29.219769	30.661956	32.071144	33.451360	34.805893

n/*m*	6	7	8	9	10	11
1	6.350541	7.239537	8.120137	8.994081	9.862598	10.726590
2	10.700188	11.729930	12.742338	13.740393	14.726326	15.701860
3	14.192298	15.300055	16.387388	17.457428	18.512597	19.554814
4	17.480827	18.642354	19.782408	20.903963	22.009383	23.100580
5	20.675303	21.876888	23.056868	24.217975	25.362428	26.492050
6	23.816716	25.049540	26.261073	27.453797	28.629756	29.790661
7	26.924753	28.182718	29.419930	30.638629	31.840691	33.027702
8	30.010256	31.288939	32.547509	33.787992	35.012101	36.221304
9	33.079761	34.375834	35.652478	36.911524	38.154539	39.382878
10	36.137463	37.448356	38.740508	40.015581	41.275010	42.520044

argument $\xi_0 > 1$ must be determined. These roots are complex roots of the form $\lambda_{mn} = -\frac{1}{2} + i\alpha^*_{mn}$ and require the associated Legendre function of complex degree λ. The numerical results are presented in Figs 5a, 5b and 5c, where the isothermal lines are given for the azimuthal planes $\varphi = \frac{\pi}{16}, \left(\frac{7\pi}{16}\right), \frac{\pi}{8}$ and $\frac{\pi}{4}$.

In the azimuthal plane $\varphi = \frac{\pi}{8}$, $\Delta\left(\frac{T-T_0}{T_1}\right)$ from streamline to streamline is 0.028 $\left(\frac{\mu-1}{8}\right)^2$, ($\mu = 1, 2, \ldots, 9$), in the plane $\varphi = \frac{\pi}{8}$ it is 0.028 $\left(\frac{\mu-1}{14}\right)^2$ ($\mu = 1, 2, \ldots;$ 15), and in the azimuthal plane $\varphi = \frac{\pi}{4}$ (symmetry plane) it is 0.038 $\left(\frac{\mu-1}{19}\right)^2$ ($\mu = 1$, $2, \ldots 20$).

In Tables 2 and 3, only the imaginary parts of the roots α^*_{mn} and β^*_{mn} are presented for $\xi_0 = 1.1$ ($\Delta\xi_0 = 0.1$) to $\xi_0 = 1.6$. More of these roots up to $\xi_0 = 5.0$ ($\Delta\xi_0 = 0.1$) are given in [12]. Differentiation of P^0_λ with respect to the degree λ yields after application of logarithmic differentiation and the ψ-function $\psi(x+1) - \psi(x) = \frac{1}{x}$ for

$$\frac{\partial P^0_{\lambda_n}}{\partial \lambda_n}(\eta_0) = \sum_{\nu=1}^{\infty} \frac{(-\lambda_n)_\nu(\lambda_n+1)_\nu}{2^\nu(\nu!)^2}(1-\eta_0)^\nu[\psi(\lambda_n+\nu+1) - \psi(\lambda_n-\nu+1)]$$

The expression $(\ldots)_\nu$ represents the Pochhammer notation, i.e. $(\alpha)_\nu = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+\nu-1)$, $(\alpha)_0 = 1$, while the logarithmic derivative ψ of the gammafunction Γ is given by

$$\psi(x) = -\gamma + (x-1)\sum_{\mu=0}^{\infty} \frac{1}{(\mu+1)(x+\mu)}$$

with γ as the Euler constant. Therefore

$$\psi(\lambda_n+\nu+1) - \psi(\lambda_n-\nu+1) = 2\sum_{\mu=0}^{\infty} \frac{\nu}{[(\lambda_n+\mu+1)^2 - \nu^2]}$$

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Zusammenfassung — Für beliebig ausgedehnte sphäroide Systeme wurde mit verschiedenen Randbedingungen eine stationäre lokale Temperaturverteilung ermittelt. Die Temperatur wurde für alle möglichen Fälle ermittelt, so auch für den Fall, in dem die Temperatur an der Hyperbeloberfläche konstant gehalten und an den ausgedehnten sphäroiden Flächen veränderlich ist bzw. umgekehrt. Für letzteren Fall mußten für Argumente größer als eins Wurzeln betreffs des Grades der zugehörigen Legendre Funktion ermittelt werden. Einige der dargestellten Lösungen wurden numerisch ermittelt und die isothermen Flächen dargestellt.

Резюме — Для произвольно вытянутых сфериодальных систем с различными граничными суловиями определено распределестационарной локальной температуры. Температура была выведена для всех возможных систем и даже для таких случаев, где температура была сохранена постоянной на гиперболических поверхностях и изменялась на вытянутых сфероидальных поверхностях или же наоборот. В последнем случае были определены причины того, что степень связанной функции Лежандра для аргументов была больше единицы. Численно оценены некоторые представленные решения и показаны изотермические поверхности.